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Note

The mean chromatic number of paths and cycles

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Abstract

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The mean chromatic number of a graph is defined. This is a measure of the expected performance of the greedy vertex-colouring algorithm when each ordering of the vertices is equally likely. In this note, we analyse the asymptotic behaviour of the mean chromatic number for the paths and even cycles, using generating function techniques.

The greedy algorithm is perhaps the best-known graph-colouring heuristic. Taking the colours to be the positive integers, it can be described as follows: Given an ordering σ of the vertex-set of the graph G, assign colours to the vertices in order, giving each vertex the first available colour (that is, the first colour which has not already been assigned to a vertex adjacent to it).

Let P_n, C_n denote the path and cycle graphs with vertex-set $\{1, 2, \ldots, n\}$. The paths and the even cycles have obvious bipartitions, the bipartition classes consisting of alternate vertices. In a sense, these graphs are canonical examples for which the greedy vertex-colouring algorithm has bad expected performance, because the bipartitions are almost always overlooked by the algorithm. In order to quantify the expected behaviour of the greedy algorithm we make the following definitions. For any

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ordering σ of the vertex-set of a graph G, denote by $\chi(G, \sigma)$ the number of colours used by the greedy algorithm to colour G when the vertices are presented in the order described by σ. The *mean chromatic number* of G is defined to be

$$\bar{\chi}(G)=\frac{1}{n!}\sum \chi(G, \sigma),$$

where the sum is over all orderings of the vertex-set. It is easily verified [1] that if $f_i(G)$ is the proportion of orderings σ such that $\chi(G, \sigma)\geqslant\chi(G)+i$ then

$$\bar{\chi}(G)=\chi(G)+f_1(G)+f_2(G)+\cdots,$$

where this sum is finite since $\chi(G, \sigma)$ is always at most one greater than the maximum degree of G. Consider now the path graphs $P_n\,(n\geqslant 3)$. For any ordering σ of the vertex-set $\{1,2,\ldots,n\}$ of P_n, $\chi(P_n, \sigma)$ is either 2 or 3. We shall denote by g_n the proportion of all orderings σ of the vertex-set such that $\chi(P_n, \sigma)=2$ and such that, in the resulting colouring of the path, vertex 1 is assigned colour 1. For the cycles, we shall denote by γ_n the proportion of orderings σ for which $\chi(C_n, \sigma)=2$. Note that $\gamma_n=0$ if n is odd. In [2] it is shown that $\gamma_{2n}=g_{2n-3}$. It follows from this that $\bar{\chi}(C_{2n})=2+f_1(C_{2n})=3-\gamma_{2n}=3-g_{2n-3}$.

Consider the generating function

$$\sum_{n=0}^{\infty} g_{2n+1}z^{2n+1}.$$

Let $h(z)$ be the complex function defined by this power series within its region of convergence. Bouwer and Star [2] find a differential equation for h which they solve to obtain

$$h(z)=\frac{\sinh z}{\cosh z-z\sinh z}.$$

Gessel [3] uses the concept of the hook decomposition of a permutation to prove this by a more combinatorial argument. We shall use this result to obtain an asymptotic expression for g_{2n+1}. First, we require a lemma concerning the radius of convergence of the generating function.

Lemma 1. *Let α denote the (unique) positive real solution of* $\cosh x=x\sinh x$. *Then α is the radius of convergence of the power series* $\sum g_{2n+1}z^{2n+1}$.

Proof. Let the radius of convergence of the series be r. Because α is a pole of h, the series cannot converge at α, and so $r\leqslant\alpha$. Pringsheim's theorem tells us that if a power series has positive coefficients, then the radius of convergence is a real singularity of the function defined by the series. This applies to give $r\geqslant\alpha$. Thus $r=\alpha$. □

Now,

$$\cosh x=x\sinh x \;\Leftrightarrow\; e^{2x}=\frac{x+1}{x-1}.$$

If $|x|<1$ and x is a real number then the left-hand side is positive, and the right-hand side negative. So $\alpha \geqslant 1$. In fact, α is approximately 1.19968.

Theorem 2. *As $n \to \infty$,*

$$g_{2n+1} \sim \frac{2}{\alpha^{2n+4}},$$

where α is the positive real solution of the equation $\cosh x = x \sinh x$.

Proof. We first show that the only zeroes of $\cosh z - z \sinh z$ on the circle of convergence of the generating function are $\pm\alpha$. Let z have modulus α. Then,

$$\cosh z = z \sinh z \iff 1 + \sum_{k=1}^{\infty} \frac{z^{2k}}{(2k)!} = \sum_{k=1}^{\infty} \frac{z^{2k}}{(2k-1)!}$$

$$\iff \sum_{k=1}^{\infty} \frac{(2k-1)}{(2k)!} z^{2k} = 1.$$

Now, putting $w = z^2$, this last equation becomes

$$\sum_{k=1}^{\infty} \frac{(2k-1)}{(2k)!} w^k = 1.$$

Let

$$f(w) = \sum_{k=1}^{\infty} \frac{(2k-1)}{(2k)!} w^k = \sum_{k=1}^{\infty} f_k w^k.$$

Then

$$|f(w)| \leqslant \alpha^2 |f_1 + f_2 w| + f_3 \alpha^6 + f_4 \alpha^8 + \cdots \leqslant |f(\alpha^2)| = 1,$$

with equality in the second of these inequalities only if

$$|f_1 + f_2 w| = |f_1 + f_2 \alpha^2|,$$

which is the case only when $w = \alpha^2$. Therefore, for such z, $|f(z)| = 1$ if and only if $z = \pm\alpha$, and hence these are the only zeroes on the circle.

Extending a technique from [4], let R be such that $R > \alpha$ and $h(z)$ has no poles other than $\pm\alpha$ in the closed disk $\bar{D}(0, R)$. Such an R exists by what we have just shown, and because the poles of h are discrete. Let $r < \alpha$ and let $C = C(0, r)$, $\bar{C} = C(0, R)$. By Cauchy's residue theorem, we have

$$g_{2n+1} = \frac{1}{2\pi i} \int_C \frac{h(z)}{z^{2n+2}} \mathrm{d}z = \frac{1}{2\pi i} \int_{\bar{C}} \frac{h(z)}{z^{2n+2}} \mathrm{d}z - \operatorname{res}(\alpha) - \operatorname{res}(-\alpha)$$

where

$$\operatorname{res}(\pm\alpha) = \operatorname{res}\left(\frac{h(z)}{z^{2n+2}}, \pm\alpha\right)$$

are the residues of the function $h(z)/z^{2n+2}$ at the poles α and $-\alpha$. Now, the poles at $\pm\alpha$ are simple, since the derivative of $\phi(z)=\cosh z - z\sinh z$ is nonzero at these points. Thus,

$$\operatorname{res}(\pm\alpha) = -\frac{\sinh(\pm\alpha)}{(\pm\alpha)^{2n+2}(\pm\alpha)\cosh(\pm\alpha)} = -\frac{\sinh\alpha}{\alpha^{2n+3}\cosh\alpha} = -\frac{1}{\alpha^{2n+4}}.$$

So,

$$\left| g_{2n+1} - \frac{2}{\alpha^{2n+4}} \right| = \left| \frac{1}{2\pi i} \int_{\bar{C}} \frac{h(z)}{z^{2n+2}}\,\mathrm{d}z \right| \leqslant \frac{M}{R^{2n+1}},$$

where M is an upper bound for the modulus of the continuous function h on the compact set $\bar{C}$. Hence,

$$g_{2n+1} = \frac{2}{\alpha^{2n+4}}(1+\psi(n)),$$

where

$$|\psi(n)| \leqslant \frac{\alpha^3 M}{2}\left(\frac{\alpha}{R}\right)^{2n+1}.$$

Now, $\psi(n)\to 0$ as $n\to\infty$, and so the required asymptotic expression for g_{2n+1} follows. □

It is easy to treat the coefficients g_{2n} analogously, using results of Bouwer and Star [2] and Gessel [3] on the power series $\sum g_{2n}z^{2n}$ and the function it defines, to obtain the result

$$g_{2n} \sim \frac{2}{\alpha^{2n+2}\cosh\alpha}.$$

If we also use the relation $1-f_1(P_{2n+1}) = g_{2n-1}$ from [2], we obtain asymptotic expressions for the mean chromatic numbers of the paths and even cycles:

Theorem 3. *As* $n\to\infty$,

$$3-\bar{\chi}(C_{2n}) \sim \frac{2}{\alpha^{2n}},$$

$$3-\bar{\chi}(P_{2n+1}) \sim \frac{2}{\alpha^{2n+2}},$$

$$3-\bar{\chi}(P_{2n}) \sim \frac{4}{\alpha^{2n+2}\cosh\alpha}.$$

In particular, all these mean chromatic numbers tend to 3 as n tends to infinity.

Proof. We have

$$3-\bar{\chi}(C_{2n})=1-f_1(C_{2n})=\gamma_{2n}=g_{2n-3},$$

$$3-\bar{\chi}(P_{2n+1})=1-f_1(P_{2n+1})=g_{2n-1},$$

$$3-\bar{\chi}(P_{2n})=1-f_1(P_{2n})=2g_{2n}.$$

The result follows from Theorem 2 and from the asymptotic expression for g_{2n} given above. □

Theorem 3 quantifies precisely the expected performance of the greedy vertex-colouring algorithm on the paths and cycles, providing asymptotic expressions for the expected number of colours in the resulting colouring.

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